International Research Project:

ANISOTROPIC DIRICHLET-TO-NEUMANN MAPS: HEAT KERNELS AND SPECTRAL THEORY

1. DESCRIPTION AND OBJECTIVES OF THE PROJECT

1.1. **Background.** To locate a brain tumor using conductivity imaging, small currents are injected into a body which acts as a volume conductor. Potential (voltage) measurements are taken on the boundary, and the task is to reconstruct the internal conductivity distribution of the body. The so-called Dirichlet-to-Neumann (DtN) map governs the relationship between the injected currents and the resulting potential measurements on the boundary. Note that both the injected currents and the potential measurement occur on the *boundary*. Then the mathematical formulation of this task is to construct the *internal* conductivity from this DtN map. The classical Calderón's inverse problem asks whether the DtN map carries all information of the conductivity inside the body (or the domain). Several properties of spectral type for the DtN map are known if the body is bounded and it has a smooth boundary. The DtN map also occurs in scattering theory, in the theory of homogenization and analysis of elliptic systems with rapidly oscillating coefficients and many other fields in mathematics. *The principal aim of this proposal* is to systematically analyse the DtN map on domains with a fairly rough boundary and general coefficients, so that we can consider the co-called anisotropic setting and deriving novel insights in the areas and subject in which the DtN map plays an important role.

It is known that the spectrum of the DtN map on the boundary is closely related to the spectrum of the elliptic operator with Robin boundary conditions on the domain. Operators with Robin boundary conditions have recently attracted much attention. A systematic development of regularity properties is, however, yet to be done. For an elliptic operator with dynamic boundary conditions (or Wentzell boundary conditions) a decomposition can be written in a matrix form with an operator with Dirichlet boundary conditions on the domain and the DtN map on the boundary. Important properties such as resolvent estimates in sectors and maximal regularity are known in the smooth case. The major limitations arise from the lack of knowledge of the DtN map on non-smooth domains.

This project will leverage the long-established experience of the participants in elliptic problems and spectral theory to understanding and solving key problems on the DtN map.

1.2. **Main objectives of the proposal.** After the general background of the previous paragraph we wish to state more precisely some objectives and expected results.

Let Ω be a bounded set of $\mathbb{R}^d$ whose boundary Γ is Lipschitz. Consider the differential map

$$Au = -\sum_{k,j=1}^{d} \partial_k (c_{kj}(x)\, \partial_j u) + Vu.$$

We assume that the matrix (c_{kj}) is elliptic and has bounded measurable entries, the potential V is bounded measurable on Ω. The associated Dirichlet-to-Neumann operator $\mathcal{N}$ is defined as follows: solve for $u \in W^{1,2}(\Omega)$ and $\varphi \in H^{\frac{1}{2}}(\Gamma)$ the Dirichlet problem:

$$Au = 0 \text{ on } \Omega, \quad \mathrm{Tr}(u) = \varphi \text{ on } \Gamma \tag{1}$$

and define

$$\mathscr{N}\varphi := \partial_n u = \sum_{k,j=1}^{d} c_{kj}(x)\partial_j u.n_k$$

as the co-normal derivative of u on Γ. The outer normal vector to Ω has coordinates $n_1,\cdots,n_d$. This operator acts, with some domain, in $L^2(\Gamma)$. It has good resolvent estimates and it is (minus) the generator of a holomorphic semigroup $(e^{-t\mathscr{N}})$ on $L^2(\Gamma)$.

Using the theory of Dirichlet forms and Sobolev embeddings on the boundary, we could prove by extrapolation that $e^{-t\mathscr{N}}$ has a kernel $K_t(x,y)$ (the heat kernel of $\mathscr{N}$) which satisfies for small time t the pointwise estimate

$$|K_t(x,y)| \le C t^{d-1} \quad \text{for } x,y \in \Gamma. \tag{2}$$

It is remarkable that this estimate is valid for any bounded domain with merely Lipschitz boundary and for bounded real measurable coefficients c_{kj}. However since we are interested in harmonic analysis of $\mathscr{N}$ it is not enough to have (2) in order to apply the Duong–McIntosh approach for singular integral operators. What we aim to obtain (and which would be optimal) is a Poisson bound:

$$|K_t(x,y)| \le C t^{-(d-1)}\left(1+\frac{|x-y|}{t}\right)^{-d}. \tag{3}$$

It is known that (3) implies a number of interesting properties such as H^∞-functional calculus or $L^p - L^q$ maximal regularity for the corresponding evolution equation.

The Poisson bound (3) is proved in [1] (see Section 3) under the assumptions that Ω is of class $C^{1,\varepsilon}$ and the coefficients c_{kj} are Hölder continuous. We believe that to better understand the Dirichlet-to-Neumann operator and related problems and applications described in the previous paragraph we have to understand its heat kernel. We describe below some problems and expected results we consider in this project.

- We aim to prove Poisson kernel bounds when the operator on Ω has non-smooth coefficients. We will understand in what sense the DtN map on the boundary determines the difference between the Neumann and the Dirichlet semigroups on Ω.
- Using methods from single layer potentials we will determine the domain of the DtN map on L^p-spaces on a domain Ω with a C^2-boundary or rougher. This domain of the DtN map is known for C^∞-domains and for the very special case where $p=2$ and the coefficients are real symmetric and Lipschitz continuous.
- We are also interested in high frequency analysis for related maps associated with the Helmholtz problem. Technically, one has to deal not only with kernels that have a singularity but those that combine singular and oscillatory behaviour.
- The characterization of the DtN domain is instrumental to deduce a Weyl's law for the DtN map on C^2-domains or rougher.
- On C^∞-domains the principal part of the DtN map is the square root of the Laplace–Beltrami operator on the boundary. We will determine on a C^2-domain the relation between the DtN map and the square root of the Laplace–Beltrami operator on the boundary. A key focus is to determine whether the difference of the DtN map and the square root of the Laplace–Beltrami operator is bounded on every L^p-space.
- Recently, D. Lannes and collaborators developed techniques to describe the domain of the DtN map on the boundary of an unbounded domain. We wish to apply their techniques to

consider Poisson bounds in this setting. The DtN map on unbounded domains plays a crucial role in the study of water waves.

- We shall also extend the results from [1] on commutators of the DtN map to the case of systems. This was already considered by Kenig–Lin–Shen who proved L^p-bounds by in a very particular case.

- We intend to study the spectrum of the elliptic operator A, endowed with Robin boundary conditions of the form $\partial_n u + \alpha u = 0$, on $L^2(\Omega)$, where $\alpha \in \mathbb{R}$ is a real parameter. The spectrum of this operator is, in a certain sense, dual to the spectrum of the DtN map and knowledge of the location and nature of the spectrum of A will deliver information on the DtN map. The spectrum of A is well understood in the case of smooth coefficients and regular domains, but much less is known in the non-smooth case.

2. Involved researchers

Our aim is to bring together researchers from three countries (France, New Zealand and Portugal) to work together on this project. Four of these researchers have already joint papers in recent years (see Section 3 below). The other colleagues who are involved will bring new expertise which will lead to new collaborations. This project will also give us the opportunity to strengthen our collaborations with other groups, for example with W. Arendt and M. Sauter from the university of Ulm in Germany. Three of us (A.F.M. ter Elst, J.B. Kennedy and E.M. Ouhabaz) have collaborations with them on topics of elliptic operators.

1. Institut de Mathématiques de Bordeaux, University of Bordeaux.
- El Maati Ouhabaz, Professor, PDE and Mathematical Physics group.
- Bernhard Haak, Maitre de Conférences, PDE and Mathematical Physics group.

2. Department of Mathematics, University of Auckland.
- A.F.M. ter Elst, Professor, Analysis unit.
- Melissa Tacy, Senior Lecturer, Analysis unit.

3. Department of Mathematics, Faculty of Sciences, University of Lisbon
- James Kennedy, Assistant Professor.

3. Some publications related to the project

Here is a list of joint papers between members of the project.

[1] **A.F.M. ter Elst, E.M. Ouhabaz**, Commutator estimates and Poisson bounds for Dirichlet-to-Neumann operators with variable coefficients *Calc. Var. Partial Differential Equations* 64 (2025), no. 2, Paper No. 58, 47 pp.
[2]**A.F.M. ter Elst, E.M. Ouhabaz**, The diamagnetic inequality for the Dirichlet-to-Neumann operator. *Bull. Lond. Math. Soc.* 54 (2022), no. 5, 1978–1997.

[3] **A.F.M. ter Elst, E.M. Ouhabaz**, Dirichlet-to-Neumann and elliptic operators on $C^{1+\kappa}$-domains: Poisson and Gaussian bounds. *J. Differential Equations* 267 (2019), no. 7, 4224–4273.
[4] **A.F.M. ter Elst, E.M. Ouhabaz**, Analyticity of the Dirichlet-to-Neumann semigroup on continuous functions. *J. Evol. Equ.* 19 (2019), no. 1, 21–31.
[5] **A.F.M. ter Elst, E.M. Ouhabaz**, Analysis of the heat kernel of the Dirichlet-to-Neumann operator. *J. Funct. Anal.* 267 (2014), no. 11, 4066–4109.
[6] **A.F.M. ter Elst, E.M. Ouhabaz**, Convergence of the Dirichlet-to-Neumann operator on varying domains. *Oper. Theory Adv. Appl.,* 250 Birkhäuser/Springer, Cham, 2015, 147–154.
[7] **A.F.M. ter Elst, E.M. Ouhabaz**, Partial Gaussian bounds for degenerate differential operators II. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 14 (2015), no. 1, 37–81.
[8] **A.F.M. ter Elst, E.M. Ouhabaz**, Partial spectral multipliers and partial Riesz transforms for degenerate operators. *Rev. Mat. Iberoam.* 29 (2013), no. 2, 691–713.
[9] **A.F.M. ter Elst, E.M. Ouhabaz**, Partial Gaussian bounds for degenerate differential operators. *Potential Anal.* 35 (2011), no. 2, 175–199.
[10] W. Arendt, **A.F.M. ter Elst, J.B. Kennedy**, M. Sauter, The Dirichlet-to-Neumann operator via hidden compactness. *J. Funct. Anal.* 266 (2014), no. 3, 1757–1786.
[11] W. Arendt, **A.F.M. ter Elst, J.B. Kennedy**, Analytical aspects of isospectral drums. *Oper. Matrices* 8 (2014), no. 1, 255–277.
[12] **B.H. Haak, E.M. Ouhabaz**, Exact observability, square functions and spectral theory. *J. Funct. Anal.* 262 (2012), no. 6, 2903–2927.
[13] **B.H. Haak**, D. Trung, **E.M. Ouhabaz**, Controllability and observability for non-autonomous evolution equations: the averaged Hautus test. *Systems Control Lett.* 133 (2019), 104524, 8 pp.
[14] **B.H. Haak, E.M. Ouhabaz**, Maximal regularity for non-autonomous evolution equations. *Math. Ann.* 363 (2015), no. 3-4, 1117–1145.